③ For $\alpha \in \Omega^1_{inv}(G)$ or equivalently $\alpha \in \mathfrak{g}_G^*$, we have, $X, Y \in \mathfrak{g}_G$

$$\underset{\Omega^2_{inv}(G)}{\underset{\cap}{d\alpha}}(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$$

b/c $(L_g)^* d = d (L_g)^*$.

$$= \underset{\text{b/c } \alpha(Y) \text{ constant}}{\underset{\uparrow}{0}} - 0 - \alpha([X,Y])$$

This is again a constant fcn b/c $[X,Y] \in \mathfrak{g}_G$.

In general, take

as left inv v.f.

$$\mathfrak{g}_G = \operatorname{span}\{X_1, \dots, X_n\} \qquad X_i \text{ basis.} \quad n = \dim G$$

$$\Omega^1_{inv}(G) = \mathfrak{g}_G^* = \operatorname{span}\{\alpha_1, \dots, \alpha_n\} \qquad \alpha_i \text{ is the dual basis of } X_i.$$

Then $[X_i, X_j] = \sum_{k=1}^{n} C_{ijk} X_k$ where $C_{ijk}$ are called structure constant of $G$ (w.r.t basis $\{X_1, \dots, X_n\}$).

Then

- $[X_i, X_j] = -[X_j, X_i] \implies c_{ijk} = -c_{jik}$

- Jacobi identity $\implies \sum_r c_{ijr}\, c_{rks} + c_{jkr}\, c_{ris} + c_{kir}\, c_{rjs} = 0$.

Moreover, both $d\alpha_i$ and $\alpha_i \wedge \alpha_j$ are left inv 2-forms,
← form a basis for all $i<j$

$$d\alpha_k = \sum_{i<j} a_{kij}\, \alpha_i \wedge \alpha_j$$

what are $a_{kij}$?

← assume $i<j$

- $d\alpha_k(X_i, X_j) = -\alpha_k([X_i, X_j]) = -\alpha_k\left(\sum_{k=1}^{n} c_{ijk} X_k\right) = -c_{ijk}$

- $\left(\sum_{i<j} a_{kij}\, \alpha_i \wedge \alpha_j\right)(X_i, X_j) = a_{kij}(\alpha_i \wedge \alpha_j)(X_i, X_j) = a_{kij}$

$c_{jik}\alpha_j \wedge \alpha_i$
$= c_{ijk}\alpha_i \wedge \alpha_j$

$\implies a_{kij} = -c_{ijk} \implies d\alpha_k = -\sum_{i<j} c_{ijk}\, \alpha_i \wedge \alpha_j \left(= -\frac{1}{2} \sum_{i,j} c_{ijk}\, \alpha_i \wedge \alpha_j\right)$

This last equation is called the Maurer-Cartan equation.

$\left( d\alpha + \frac{1}{2}[\alpha,\alpha] = 0 \right)$ This is closely related to the curvature of connection on a principal bundle.

<u>Rmk</u> Historically, element $\alpha \in \Omega^1_{\mathrm{inv}}(G)$ is called a Maurer-Cartan form.

Recall that we learned connection of a vector bundle $\begin{smallmatrix} E \\ \downarrow \\ M \end{smallmatrix}$, denoted by $\nabla$. For $E = TM$, where $M = (M, g)$ a Riem mfd, there exists a unique connection $\nabla$ on $\begin{smallmatrix} TM \\ \downarrow \\ M \end{smallmatrix}$ that is compatible with the Riem metric and torsion-free:

Levi-Civita connection

(i) $Z\, g(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \quad \forall X, Y, Z \in \Gamma(TM)$

(ii) $\nabla_X Y - \nabla_Y X = [X, Y] \quad \forall X, Y \in \Gamma(TM)$ ← torsion-free

($\underset{\text{PF}}{\Longrightarrow}$ Koszul formula: $2g(\nabla_X Y, Z) = X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X)$)

**Rmk** torsion and curvature are two independent quantities.

Fix a basis vector fields $\{X_1, \cdots, X_n\}$ of $TM$, then for any $X \in \Gamma(TM)$

The existence of such basis implies that $TM$ is trivial. This holds for some mfds, NOT ALL! For instance, any Lie group $G$ has $TG$ trivial.

$$\underset{\Gamma(TM)}{\underset{\cap}{\nabla_X X_i}} = \sum_{j=1}^n \theta_i^j(X)\, X_j$$

where $\theta_i^j : \Gamma(TM) \to \mathbb{R}$ a $C^\infty(M)$-linear map, so $\theta_i^j \in \Gamma(T^*M)$ is a 1-form, called the <u>connection form</u>.

$$\nabla_{fX} X_i = \sum_{j=1}^n \theta_i^j(fX)\, X_j$$

$$= \sum_{j=1}^n f\, \theta_i^j(X)\, X_j$$

$$(\Rightarrow \theta_i^j(fX) = f\, \theta_i^j(X))$$

Observations

- $\{\theta_i^j\}_{1 \le i,j \le n}$ forms a matrix $\theta = (\theta_i^j)_{1 \le i,j \le n}$.

  called connection matrix (w.r.t basis $X_1, \cdots, X_n$)

- Assume $\{X_1, \cdots, X_n\}$ are orthonormal (we can do it by Gram-Schmit process)

  then $\theta$ is a skew-symmetric matrix (i.e. $\theta^T = -\theta$).

$$g(X_i, X_j) = \begin{cases} 1 & i = j \\ 0 & i \ne j \end{cases}$$

Indeed, $\quad 0 = X g(X_i, X_j)$

for any $i,j$

$$= g(\nabla_X X_i, X_j) + g(X_i, \nabla_X X_j)$$

$$= g\Big(\sum_{k=1}^n \theta_i^k(X) X_k, X_j\Big) + g\Big(X_i, \sum_{k'=1}^n \theta_j^{k'}(X) X_{k'}\Big)$$

$$= \sum_{k=1}^n g(\theta_i^k(X) X_k, X_j) + \sum_{k'=1}^n g(X_i, \theta_j^{k'}(X) X_{k'})$$

take $k=j$

and $k'=i$

$$= \sum_{k=1}^n \theta_i^k(X) g(X_k, X_j) + \sum_{k'=1}^n \theta_j^{k'}(X) g(X_i, X_{k'})$$

$$= \theta_i^j(X) + \theta_j^i(X) \quad \checkmark$$

Recall curvature $\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} =: R(X,Y)$

Then under a basis $\{X_1, \cdots, X_n\}$. $\leftarrow$ not nec orthonormal

$$R(X,Y) X_i = \sum \Omega_i^j(X,Y) X_j.$$

which defines a 2-tensor $\Omega_i^j(X,Y) \leftarrow$ need to verify (DIY)

$\Omega_i^j(fX, gY) = fg\, \Omega_i^j(X,Y)$

Let's do some computation.

$$\nabla_X \nabla_Y X_i = \nabla_X \left( \sum_{k=1}^{n} \theta_i^k(Y) X_k \right)$$

$$= \sum_{k=1}^{n} \nabla_X \left( \theta_i^k(Y) X_k \right) = \sum_{k=1}^{n} \underbrace{X(\theta_i^k(Y))}_{\text{directional derivative}} X_k + \theta_i^k(Y) \nabla_X X_k$$

$$= \sum_{k=1}^{n} X(\theta_i^k(Y)) X_k + \left( \theta_i^k(Y) \sum_{l=1}^{n} \theta_k^l(X) X_l \right)$$

Switching $Y$ and $X$, we get

$$\nabla_Y \nabla_X X_i = \sum_{k=1}^{n} Y(\theta_i^k(X)) X_k + \sum_{k=1}^{n} \theta_i^k(X) \sum_{l=1}^{n} \theta_k^l(Y) X_l .$$

$$\nabla_{[X,Y]} X_i = \sum_{k=1}^{n} \theta_i^k([X,Y]) X_k$$

$$\Rightarrow R(X,Y) X_i = \sum_{k=1}^{n} \Big( \underbrace{X(\theta_i^k(Y)) - Y(\theta_i^k(X)) - \theta_i^k([X,Y])}_{\text{PART 1}} \Big) X_k$$

$$+ \sum_{l=1}^{n} \Big( \sum_{k=1}^{n} \underbrace{\theta_i^k(Y)\theta_k^l(X) - \theta_i^k(X)\theta_k^l(Y)}_{\text{PART 2}} \Big) X_l$$

$$\left.\begin{aligned} \underline{\text{PART 1}} &= d\theta_i^k(X,Y) \\ \underline{\text{PART 2}} &= (\theta_k^l \wedge \theta_i^k)(X,Y) \end{aligned}\right\} \Rightarrow R(X,Y)X_i = \left(\sum_{k=1}^n d\theta_i^k + \sum_{k,l=1}^n \theta_k^l \wedge \theta_i^k\right)(X,Y)X_k$$

So for fixed $X_j$ ($k=j$ and $l=k$), we have

$$\Omega_i^j = d\theta_i^j + \sum_{k=1}^n \theta_k^j \wedge \theta_i^k$$

Moreover, under an orthonormal basis, we have

$$\begin{aligned} \Omega_i^j &= d\theta_i^j + \sum_{k=1}^n \theta_k^j \wedge \theta_i^k \\ &= d(-\theta_j^i) + \sum_{k=1}^n (-\theta_k^j) \wedge (-\theta_i^k) \\ &= -d\theta_j^i - \sum_{k=1}^n \theta_k^i \wedge \theta_j^k \quad \leftarrow \text{Here, we switch the position } \theta_k^i \text{ and } \theta_j^k. \\ &= -\Omega_j^i \end{aligned}$$

So if we form a matrix $\Omega = (\Omega_i^j)_{1\le i,j\le n}$, then it is also a skew-symmetric matrix.

Moving one step forward, we don't like the output to be a vector field, instead we like the output to be a number.

$$\underbrace{R(X,Y)}_{\text{a vector field}} Z \rightsquigarrow \text{a number}$$

Define the $(0,4)$-tensor

$$R(X,Y,Z,W) := g(R(X,Y)Z, W) \quad \text{for } X,Y,Z,W \in \Gamma(TM)$$

<u>Exe</u>. Prove $R(X,Y,Z,W)$ satisfies the following properties.

- $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$
- $R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z)$
- $R(X,Y,Z,W) = R(Z,W,X,Y)$

$\Rightarrow$ Sectional curvature: for a 2-dim'l subspace $P \subseteq T_xM$, define <u>sectional curvature at pt $x \in M$ of $P$</u> by

$$K(p) := R(X_1, X_2, X_2, X_1)(x) \leftarrow$$

for an arbitrary basis $\{X, Y\}$ of $P$,

$$K(p) := \frac{R(X,Y,X,Y)}{g(X,X)\,g(Y,Y) - g(X,Y)^2}$$

for any orthonormal basis $\{X_1, X_2\}$ of $P$.

Rmk $K(p)$ is independent of the choice of orthonormal basis $\{X_1, X_2\}$.

Rmk When $\dim M = 2$, $K(p) = K(T_xM) \leftarrow$ curvature function on $M$. (related/equal to Gauss curvature)

Example $\mathbb{H}^2_+ = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ where $g(-,-) = \dfrac{dx \otimes dx + dy \otimes dy}{y^2}$.

(cf. HW4). Choose an orthonormal basis

$$X_1 = y\frac{\partial}{\partial x} \quad \text{and} \quad X_2 = y\frac{\partial}{\partial y}$$

$g(X_1, X_1) = g(X_2, X_2) = 1$ and $g(X_1, X_2) = 0$. Under this basis $\{X_1, X_2\}$, the connection matrix is (defined by)

$$\theta = \begin{pmatrix} 0 & -\frac{1}{y}dx \\ \frac{1}{y}dx & 0 \end{pmatrix} \quad \Big(\text{so } \theta^1_2 = -\underbrace{\tfrac{1}{y}dx}_{\text{1-form}}\Big)$$

Then

$$\underbrace{\Omega^1_2}_{\text{the only non-trivial entry}} = d\theta^1_2 + \sum_{k=1}^{2} \theta^k_2 \wedge \theta^1_k = d\theta^1_2 = -\frac{1}{y^2}\,dx\wedge dy$$

Therefore the sectional curvature at pt $(x,y)$ is

$$\begin{aligned} R(X_1, X_2, X_2, X_1) &= g\big(R(X_1,X_2)X_2, X_1\big) \\ &= g\Big(\sum_{j=1}^{2} \Omega^j_2(X_1,X_2)X_j, X_1\Big) \\ &= g\big(\Omega^1_2(X_1,X_2)X_1, X_1\big) \\ &= \Omega^1_2(X_1,X_2) = -\frac{1}{y^2}\,dx\wedge dy\Big(y\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\Big) = -1 \end{aligned}$$

<u>Rmk</u> Some book define $K(p)$ using $R(X_1, X_2, X_1, X_2)$, but it defines $R(X,Y)$ by $\nabla_Y\nabla_X - \nabla_X\nabla_Y + \nabla_{[X,Y]}$ (the opposite of our definition!).

Let's get back to Lie group.

Here is a basic observation: Any Lie group $G$ admits a left inv. Riem metric denoted by $\langle , \rangle$, i.e., $(L_g)^* \langle , \rangle = \langle , \rangle$ or more explicitly,

$$(L_g)^* \langle v, w \rangle_x = \langle (L_g)_*(v), (L_g)_*(w) \rangle_{gx} \quad \text{for any } v, w \in T_x G.$$

Construction: pick any inner product $\langle , \rangle$ on $T_e G$, then define

$$\langle -, - \rangle_x := \langle (L_{x^{-1}})_*(-), (L_{x^{-1}})_*(-) \rangle.$$

What's non-trivial is ~~that~~:

<u>Prop</u> Any cpt Lie group $G$ admits a <u>bi-invariant</u> metric

$$(L_g)^* \langle , \rangle = \langle \rangle \text{ and } (R_g)^* \langle , \rangle = \langle \rangle.$$

where $R_g$ means right multiplication by $g$.

* One usually write $R_{g^{-1}}$ (instead of $R_g$) – b/c $R_{g^{-1}} \circ L_g = c(g)$ conjugation by $g$.

<u>pf</u>. Assume $\dim G = n$.

<u>Step 1.</u> Pick a basis $\{v_1, \cdots, v_n\}$ of $T_eG$ and consider 1-forms

$$\alpha_i = \text{dual of } X^{v_i} \qquad (\text{so } \alpha_i \in \Omega^1_{inv}(G))$$

Then $\alpha_1 \wedge \cdots \wedge \alpha_n$ is a nowhere vanishing $n$-form on $G$.

a volume form

Indeed, $\forall g \in G$

$$(\alpha_1 \wedge \cdots \wedge \alpha_n)_g (X^{v_1}(g), \cdots, X^{v_n}(g)) = (\alpha_1 \wedge \cdots \wedge \alpha_n)_g ((L_g)_*(v_1), \cdots, (L_g)_*(v_n))$$

$$= (L_g)^*(\alpha_1 \wedge \cdots \wedge \alpha_n)_e (v_1, \cdots, v_n) = 1.$$

(Note that, in particular, any Lie group is orientable!)

Similarly, one can construct a volume form $\Omega$ on $G$ that is <u>right</u> invariant.

<u>Step 2.</u> Pick any inner product $\langle , \rangle_e$ at $T_eG$, which induces

a $\underline{\underline{\text{right}}}$ inv metric $\langle , \rangle$ on $G$.

Define for $X, Y \in T_g G$,

for fixed $g \in G$, this is a function with variable $a \in G$

$$\langle\langle X, Y \rangle\rangle_g := \int_{a \in G} \overbrace{\langle (L_a)_* X, (L_a)_* Y \rangle_{ag}} \, \Omega$$

averaging trick (very common in rep theory)

$\underline{\text{Step 3}}$ $\langle\langle , \rangle\rangle$ is left inv. For any $h \in G$

$$\begin{aligned}
(L_h)^* \langle\langle X, Y \rangle\rangle_g &= \langle\langle (L_h)_* X, (L_h)_* Y \rangle\rangle_{hg} \\
&= \int_G \langle ((L_a)_* \cdot (L_h)_*) X, ((L_a)_* \cdot (L_h)_*) Y \rangle_{ahg} \, \Omega \\
&= \int_G \langle (L_{ah})_* X, (L_{ah})_* Y \rangle_{(ah)g} \, \Omega \\
\overset{\text{key step}}{=} &\int_{\underset{(=G)}{R_h \cdot G}} F \cdot \Omega
\end{aligned}$$

where $F(a) = \langle (L_a)_* X, (L_a)_* Y \rangle_{ag}$ as a function on $G$

base change formula $= \int_G R_h^*(F \cdot \Omega) = \int_G (F \circ R_h) \cdot \Omega$

b/c $\Omega$ is right invariant

precisely due to the averaging trick $= \int_G F \cdot \Omega$

$$= \int_G \langle (L_a)_* X, (L_a)_* Y \rangle_{ag} = \langle\langle X, Y \rangle\rangle.$$

<u>Step 4</u> $\langle\langle\ \rangle\rangle$ is right inv. For any $h \in G$,

$$(R_h)^* \langle\langle X, Y \rangle\rangle_g = \langle\langle (R_h)_* X, (R_h)_* Y \rangle\rangle_{gh}$$

$$= \int_G \langle (L_a)_* (R_h)_* X, (L_a)_* (R_h)_* Y \rangle_{agh} \Omega$$

$L_a, R_h$ commute. $= \int_G R_h^* \langle (L_a)_* X, (L_a)_* Y \rangle_{ag} \Omega$

$$= \int_G \langle (L_a)_* X, (L_a)_* Y \rangle_{ag} \Omega = \langle\langle X, Y \rangle\rangle_g. \quad \square$$

eg To construct a bi-inv metric on $SU(n) = \{A \in U(n) \mid \det(A) = 1\}$

one can start from an inner product on its Lie algebra

$$su(n) = \{ X \in Gl(n,\mathbb{C}) \mid X^* + X = 0 \text{ and } tr(X) = 0 \}$$

defined by

$$\langle X, Y \rangle = Re\, tr(XY^*).$$

a general Lie group, not nec. compact

**Thm** (Milnor) A Lie group $G$ admits a bi-invariant iff $G$ is isomorphic to $N \times H$ where $N$ cpt and $H$ is abelian.

Next, we will see the existence of a bi-inv metric (even left inv) on $G$ considerably simplifies (geometric) computation on $G$.

e.g. Given $(G, g)$ where $g$ is left inv. and $X, Y$ are left inv v.f. on $G$, then $g(X,Y) \equiv$ constant.

In particular, if $\nabla$ is a Levi-Civita metric on $(G,g)$, then Koszul formula simplifies to:

$$2g(\nabla_X Y, Z) = g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X). \qquad (*)$$

Recall the adjoint rep (of Lie algebra) $ad : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}) =: T_{1} GL(\mathfrak{g})$.

we have $ad_X(Y) = [X,Y]$. Therefore

a linear transformation on $\mathfrak{g}$

$$(*) \iff g(\nabla_X Y, Z) = \frac{1}{2}\Big(g([X,Y],Z) - \underbrace{g(ad_X(Z), Y)}_{= g(Y,\, ad_X(Z)) = g((ad_X)^* Y, Z)} - \underbrace{g(ad_Y(Z), X)}_{= g((ad_Y)^* X, Z)}\Big)$$

$g(X, AY) = g(A^*X, Y)$ where $A^*$ is called the adjoint operator of $A$ with respect to a non-deg bilinear form (here is $g$).

$$\iff g(\nabla_X Y, Z) = \frac{1}{2}\, g\big([X,Y] - (ad_X)^* Y - (ad_Y)^* X,\ Z\big)$$

$$\implies \nabla_X Y = \frac{1}{2}\big([X,Y] - (ad_X)^* Y - (ad_Y)^* X\big). \qquad (\heartsuit)$$

Moreover, if $g$ is bi-inv, then

$$(ad_X)^* = -ad_X.$$

Indeed, by def,

$$(\mathrm{ad}_X)(Y) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\exp(tX)}(Y)$$

Therefore since $\langle \mathrm{Ad}_{\exp(tX)}(Y), \mathrm{Ad}_{\exp(tX)}(Z) \rangle = \langle Y, Z \rangle$, then

pushforward of multiplication by $\exp(tX)$
on both left and right (its inverse)

$$\left\langle \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\exp(tX)}(Y), Z \right\rangle + \left\langle Y, \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\exp(tX)}(Z) \right\rangle = 0.$$

$$\Longleftrightarrow \quad \langle (\mathrm{ad}_X)(Y), Z \rangle = -\langle Y, (\mathrm{ad}_X)(Z) \rangle$$

$$\Rightarrow \quad g((\mathrm{ad}_X)(Y), Z) = g(-(\mathrm{ad}_X)^*(Y), Z)$$

$$\Rightarrow \quad \mathrm{ad}_X = -(\mathrm{ad}_X)^*$$

<u>Prop</u> On $(G, g, \nabla)$ where $g$ is a bi-inv metric and $\nabla$ is the Levi-Civita connection of $(G, g)$, we have $\nabla_X Y = \frac{1}{2}[X, Y]$.

Pf $\quad \nabla_X Y = \frac{1}{2}\Big([X,Y] + \underbrace{\cancel{(\mathrm{ad}_X)^*(Y)} + \cancel{(\mathrm{ad}_Y)^*(X)}}_{\text{cancelled}}\Big) = \frac{1}{2}[X,Y]. \quad \square$

<u>Rmk</u> In particular, $X = Y \Rightarrow \nabla_X X = \frac{1}{2}[X,X] = 0.$ $\leftarrow$ therefore, the integral curve of a left inv v.f. under bi-inv metric $g$ on $G$ is a <u>geodesic</u> 测地线.

How about curvature?

From ($\nabla$), observe that if $X, Y$ are left inv. v.f. then $\nabla_X Y$ is also a left inv. v.f. Therefore

$$Y\, g(\nabla_X Z, W) = 0 \quad \text{for all left inv input v.fs.}$$

$$\Rightarrow \quad 0 = Y\, g(\nabla_X Z, W)$$

$$\xrightarrow{\text{defining axiom}} \; = g(\nabla_Y \nabla_X Z, W) + g(\nabla_X Z, \nabla_Y W)$$

$$\Rightarrow \quad g(\nabla_Y \nabla_X Z, W) = -g(\nabla_X Z, \nabla_Y W).$$